Dispersion relation for electromagnetic wave propagation in a strongly magnetized plasma

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Abstract. A dispersion relation for electromagnetic wave propagation in a strongly magnetized cold plasma is deduced, taking photon–photon scattering into account. It is shown that the combined plasma and quantum electrodynamic effect is important for understanding the mode-structures in magnetar and pulsar atmospheres. The implications of our results are discussed.

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1. Introduction

The quantum electrodynamical (QED) phenomenon of elastic photon–photon scattering, due to the interaction of photons with virtual electron–positron pairs, has recently received increased attention [1–11]. Several papers are motivated by the desire to detect photon–photon scattering in laboratories [1–4], whereas others [5–7] concern phenomena that might be relevant when the laser power is further increased to produce electric fields strengths close to the Schwinger field $\sim 10^{18}\,\mathrm{V\,m^{-1}}$ [7]. Up to now, however, observable effects of photon–photon scattering are likely to occur only for astrophysical systems [9,10,12,15–18], where the large magnetic field strengths in pulsar and magnetar environments [12–14] open up for QED processes to play an important role, leading to phenomena such as frequency down-shifting [15, 16] and lensing [17]. The frequency down-shifting is a result of so called photon splitting [15, 16], which is one of the consequences of elastic photon-photon scattering, and the process may even be responsible for the radio silence of magnetars [18]. Another QED-process of interest in pulsar and magnetar environments is pair-production [19] due to the strong field interactions, which lead to the presence of an electron-positron pair plasma in the pulsar and magnetar atmospheres. However, with a few exceptions (e.g. [8, 10]), we note that most papers considering photon–photon scattering have omitted plasma effects when considering electromagnetic wave propagation under these conditions.

In the present paper we will consider electromagnetic wave propagation at an arbitrary angle to a strong external magnetic field $\mathbf{B}_0$, and include the QED-effects associated with that field, as well as the influence of an electron-positron pair plasma.

The former effect is described within the framework of the Heisenberg–Euler Lagrangian, which constitutes an effective theory of photon–photon scattering [20, 21], and the latter contribution follows from elementary plasma theory. A comparatively general dispersion relation will be derived. It reduces to previous results in a number of limiting cases [10, 16, 22]. In order to determine the contribution from the pair-plasma on the propagation properties in pulsar and magnetar atmospheres, we adopt the Goldreich-Julian expression for the plasma density [19], and evaluate the dispersion relation for field strengths in the pulsar and magnetar range, $B_0 \sim 10^8 - 10^{10}\,\mathrm{T}$. In the radio-wave regime it then turns out that for one of the EM-wave polarizations, the plasma effects are typically negligible as compared to the QED-effects, whereas for the other polarization, the opposite is true in most cases. Noting that important processes in pulsar and magnetar environments, e.g. photon splitting, typically involve both EM-wave polarizations, we will conclude that QED and plasma effects should be simultaneously included when studying radio wave propagation in such environments.

2. Derivations

If vacuum fluctuations are taken into account, such as under highly energetic conditions (e.g. pulsar plasmas and the next generation of laser-plasma systems), Maxwell's equation will be altered by the quantum vacuum self-interaction through the polarization

$$\mathbf{P} = 2\kappa\epsilon_0^2\left[2(E^2 - c^2B^2)\mathbf{E} + 7c^2(\mathbf{E}\cdot\mathbf{B})\mathbf{B}\right] \tag{1}$$

and magnetization

$$\mathbf{M} = 2\kappa\epsilon_0^2 c^2\left[-2(E^2 - c^2B^2)\mathbf{B} + 7(\mathbf{E}\cdot\mathbf{B})\mathbf{E}\right] \tag{2}$$

respectively, see e.g. [3]. Here $\kappa = (\alpha/90\pi)(1/\epsilon_0 E_{\mathrm{crit}}^2)$ gives the strength of the quantum vacuum nonlinearity, $\alpha \approx 1/137$ is the fine-structure constant, $E_{\mathrm{crit}} = m^2c^3/e\hbar \sim 10^{18}\,\mathrm{V\,m^{-1}}$ is the Schwinger critical field, m is the electron rest mass, c is the speed of light in vacuum, e is the magnitude of the electron charge, and $\hbar$ is Planck's constant divided by 2π. These corrections to Maxwell's vacuum equations are valid as long as $|\mathbf{E}| \ll E_{\mathrm{crit}}$ and $\omega \ll \omega_e = mc^2/\hbar$, where $\omega_e \approx 8\times10^{20}\,\mathrm{rad\,s^{-1}}$ is the Compton frequency.

Next we Fourier decompose the electromagnetic perturbations, which have frequencies ω and wavevectors $\mathbf{k}$. Maxwell's equations together with the plasma equations of motion then yield

$$\Delta^{ab}\delta E_b = 0. \tag{3}$$

using index notation. Here the matrix $\Delta^{ab} = n^a n^b - n^2\delta^{ab} + \epsilon^{ab}$, where $n^a = k^a c/\omega$, $n = kc/\omega$ is the plasma refractive index, where $k = |\mathbf{k}|$, $\epsilon^{ab} = \epsilon^{ab}_{\mathrm{classical}} + \epsilon^{ab}_{\mathrm{QED}}$ is the dielectric tensor,

$$\epsilon^{ab}_{\mathrm{classical}} = \delta^{ab} + i\omega\sum_s\left(\frac{\omega_{\mathrm{p}s}}{\omega}\right)^2\sigma_s^{ab}, \tag{4}$$

$$\epsilon^{ab}_{\mathrm{QED}} = -4\xi\left[\delta^{ab} + n^a n^b - n^2\delta^{ab} - \frac{7}{2}b^a b^b - 2(\eta^{aij}n_i b_j)(\eta^{bkl}n_k b_l)\right], \tag{5}$$

s denotes the plasma particle species, $\omega_{\mathrm{p}s} = (q_s^2 n_s/\epsilon_0 m_s)^{1/2}$ is the plasma frequency for species s, $\xi = \kappa\epsilon_0 c^2 B_0^2 = (\alpha/90\pi)(cB_0/E_{\mathrm{crit}})^2$ is the dimensionless QED parameter, $b^a = B_0^a/B_0$, and

$$(\sigma_s^{ab})^{-1} = -i\omega\delta^{ab} + \omega_{\mathrm{c}s}\eta^{abj}b_j, \tag{6}$$

with the cyclotron frequency $\omega_{\mathrm{c}s} = q_s B_0/m_s$ for species s, δ^{ab} is the Kronecker dela and η_{abc} is the totally anti-symmetric unit tensor. From the definition (6) we obtain

$$\sigma_s^{ab} = \frac{i\omega}{\omega^2 - \omega_{\mathrm{c}s}^2}(\delta^{ab} - b^a b^b) - \frac{\omega_{\mathrm{c}s}}{\omega^2 - \omega_{\mathrm{c}s}^2}\eta^{abj}b_j + \frac{i}{\omega}b^a b^b, \tag{7}$$

and the dielectric tensor (4) is thus

$$\epsilon_{\mathrm{classical}}^{ab} = \delta^{ab} - \sum_s \left[\frac{\omega_{\mathrm{p}s}^2}{\omega^2 - \omega_{\mathrm{c}s}^2}(\delta^{ab} - b^a b^b) + \frac{i\omega_{\mathrm{p}s}^2\omega_{\mathrm{c}s}}{\omega(\omega^2 - \omega_{\mathrm{c}s}^2)}\eta^{abj}b_j + \left(\frac{\omega_{\mathrm{p}s}}{\omega}\right)^2 b^a b^b\right], \tag{8}$$

We note that the full dielectric tensor depends on the wavevector through the QED contribution $\epsilon_{\mathrm{QED}}^{ab}$. Freely propagating waves are characterized by the vanishing of the dispersion relation $D(\omega, \mathbf{k}) = \det(\Delta^{ab})$. Writing the QED-tensor $\epsilon_{\mathrm{QED}}^{ab}$in matrix form, letting the $\mathbf{k}$-vector lie in the xz-plane, we then obtain

$$\epsilon_{\mathrm{QED}}^{ab} = -4\xi \begin{pmatrix} 1 - n_\parallel^2 & 0 & n_\perp n_\parallel \\ 0 & 1 - n^2 - 2n_\perp^2 & 0 \\ n_\perp n_\parallel & 0 & -\frac{5}{2} - n_\perp^2 \end{pmatrix} \tag{9}$$

where $n_\parallel = k_\parallel c/\omega$, $n_\perp = k_\perp c/\omega$ and the $\mathbf{k}$-vector is written as $\mathbf{k} = k_\perp\widehat{\mathbf{x}} + k_\parallel\widehat{\mathbf{z}}$. From (8) the classical contributions to Δ^{ab} is

$$\begin{pmatrix} 1 - \sum_s \frac{\omega_{\mathrm{p}s}^2}{\omega^2 - \omega_{\mathrm{c}s}^2} - n_\parallel^2 & i\sum_s \frac{\omega_{\mathrm{p}s}^2\omega_{\mathrm{c}s}}{\omega(\omega^2 - \omega_{\mathrm{c}s}^2)} & n_\perp n_\parallel \\ -i\sum_s \frac{\omega_{\mathrm{p}s}^2\omega_{\mathrm{c}s}}{\omega(\omega^2 - \omega_{\mathrm{c}s}^2)} & 1 - \sum_s \frac{\omega_{\mathrm{p}s}^2}{\omega^2 - \omega_{\mathrm{c}s}^2} - n^2 & 0 \\ n_\perp n_\parallel & 0 & 1 - \sum_s \frac{\omega_{\mathrm{p}s}^2}{\omega^2} - n_\perp^2 \end{pmatrix} \tag{10}$$

The determinant of the sum of the matrixes (9) and (10) is then evaluated to give the dispersion relation

$$\begin{aligned} 0 = &\left((1 - n^2)(1 - 4\xi) - \sum_s \frac{\omega_{\mathrm{p}s}^2}{\omega^2 - \omega_{\mathrm{c}s}^2} + 8\xi n_\perp^2\right) \times \\ &\left[\left((1 - n_\parallel^2)(1 - 4\xi) - \sum_s \frac{\omega_{\mathrm{p}s}^2}{\omega^2 - \omega_{\mathrm{c}s}^2}\right)\left(1 + 10\xi - \sum_s \frac{\omega_{\mathrm{p}s}^2}{\omega^2} - n_\perp^2\,(1 - 4\xi)\right) - n_\perp^2 n_\parallel^2\,(1 - 4\xi)\right] - \\ &\left(\sum_s \frac{\omega_{\mathrm{p}s}^2\omega_{\mathrm{c}s}}{\omega(\omega^2 - \omega_{\mathrm{c}s}^2)}\right)^2\left(1 + 10\xi - \sum_s \frac{\omega_{\mathrm{p}s}^2}{\omega^2} - n_\perp^2\,(1 - 4\xi)\right) \end{aligned} \tag{11}$$

The dispersion relation (11) is the main result of the present paper. It describes wave propagation at any angle to the external magnetic field in a multi-component plasma, and it includes the QED effects associated with the external magnetic field.

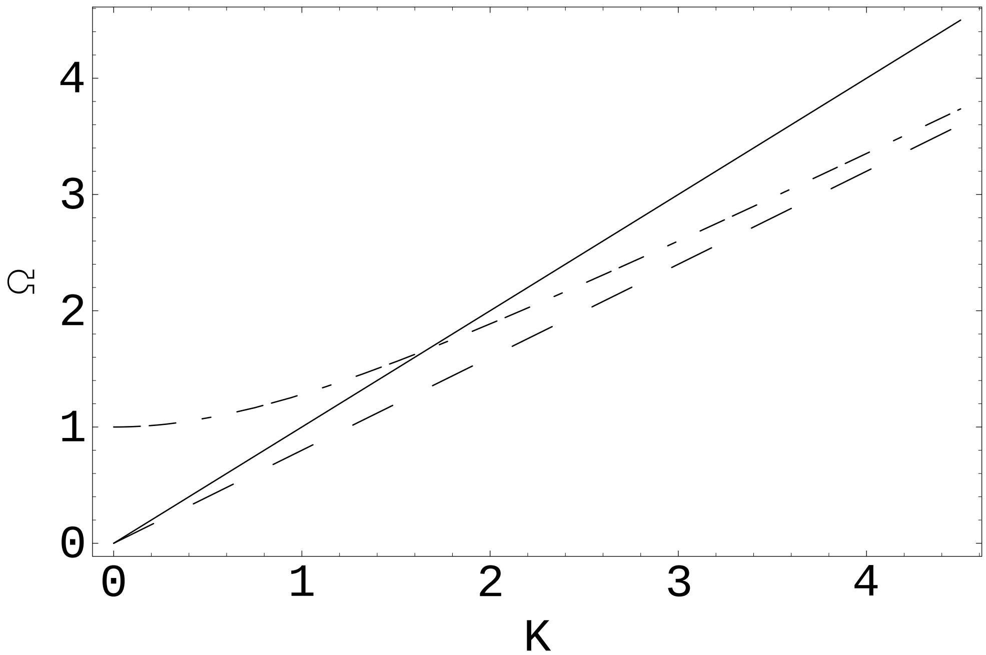

Figure 1. The normalized frequency $\Omega = \omega/\omega_{\mathrm{p}}$ plotted as a function of the normalized wavenumber $K = ck/\omega_{\mathrm{p}}$ for three cases where $k_z = 0$. The full line represents the vacuum dispersion curve, the dashed line represents the strong magnetic field QED corrected vacuum dispersion, and the dashed-dotted line depicts the combined effect of plasma dispersion and QED effects. The QED corrections corresponds to a magnetic field strength of the order of 10^{10} T, i.e. field strengths expected around magnetars.

Thus, it applies to high frequency electromagnetic waves of any polarization, as well as electrostatic oscillations and low frequency waves, such as Alfvén waves. As a specific example of how the plasma dispersion relation is affected by the QED effects we consider the case of an electron–positron plasma with $\omega \sim \omega_{\mathrm{p}} \ll |\omega_{\mathrm{c}}|$. The dispersion relation relation for the ordinary mode propagating perpendicular to the background magnetic field, with strength $\sim 10^{10}$ T is depicted in figure 1. A number of limiting cases of (11) have previously appeared in the literature. First, neglecting the QED-effects (i.e. letting $\xi \to 0$), we immediately obtain the standard dispersion relation for a cold multi-component plasma (see e.g. [22]). Alternatively, letting $\omega_{\mathrm{p}} \to 0$, we note that the dispersion relation depends on the propagation angle relative to the magnetic field. Furthermore, we note that the indicies of refraction depend on the polarization even without a plasma. These QED-effects due to a strong external magnetic field are wellknown (often referred to as "birefringence of vacuum"). Our dispersion relation in the limit $\omega_{\mathrm{p}} \to 0$ agrees with those of previous works, see e.g. [15,16]. The combined contribution from the QED-effects due to a strong magnetic field and a non-zero plasma density have previously been considered [10] in the limit of parallel propagation and allowing for large amplitudes. Taking the limit of a small wave amplitudes in the dispersion relation (11) of reference [10], and letting $n_{\perp} \to 0$ in (11) we obtain agreement with [10].

3. Conclusion

QED-effects associated with the external magnetic field are likely to be of importance in environments with extreme magnetic fields, in particular in the vicinity of astrophysical objects like pulsars and magnetars. For example, the radio silence of magnetars is assumed to be connected with QED-effects associated with the magnetar fields [18], which could reach $10^{10}-10^{11}$ T close to the surface. However, in the same environments, we also expect the presence of an electron-positron plasma [19]. Thus we evaluate (11) with $\sum_s = \sum_{e,p}$, where e and p denotes electrons and positrons, respectively. Considering propagation at an arbitrary angle to the magnetic field in an electron-positron plasma, letting $\omega_{\mathrm{pe,pp}} \sim \omega \ll |\omega_{\mathrm{ce},cp}|$, using $\xi \ll 1$ and noting that the factor $[\sum_{e,p} \omega_{\mathrm{p}s}^2 \omega_{\mathrm{c}s}/\omega(\omega^2 - \omega_{\mathrm{c}s}^2)]^2$ then becomes negligibly small (due to the approximate cancellation of the electron and positron contributions), we find from (11) that the dispersion relation separates in two modes that can be approximated by

$$1 - n^2 + 8\xi n_\perp^2 + \frac{\omega_{\mathrm{p}}^2}{\omega_{\mathrm{c}}^2(1-4\xi)} \approx 0 \tag{12}$$

and

$$(1-n^2)(1-4\xi) - \left(-14\xi + \frac{\omega_{\mathrm{p}}^2}{\omega^2}\right)(1-n_\parallel^2) \approx 0, \tag{13}$$

where $\omega_{\mathrm{p}} = (\omega_{\mathrm{pe}}^2 + \omega_{\mathrm{pp}}^2)^{1/2}$ is the total plasma frequency, and $\omega_{\mathrm{c}} = eB_0/m$ is the magnitude of the electron (or positron) cyclotron frequency. In the vicinity of pulsars or magnetars where $\omega_{\mathrm{c}} \sim 10^{19}$–$10^{21}\,\mathrm{rad\,s^{-1}}$, the last term of (12) is negligible unless the plasma density is extremely high. Omitting that term, the dispersion relation (12) is then the same as that used in [16] for the high phase velocity mode when considering photon-splitting. Similarly the ordinary mode described by (13), reduces to the mode with the lower phase velocity of reference [16] when the plasma is removed. However, for the latter dispersion relation we note that a relatively modest plasma density is enough to significantly affect the propagation properties in the radio wave regime. To make a concrete estimate, we adopt the Goldreich–Julian density

$$n_{\mathrm{GJ}} = 7\times 10^{15}\left(\frac{0.1}{\tau}\right)\left(\frac{B_{\mathrm{pulsar}}}{10^8}\right)\,\mathrm{m}^{-3} \tag{14}$$

where τ is the pulsar period time (in seconds) and B_{pulsar} the pulsar magnetic field (in tesla). The pair plasma density is expected to satisfy $n_e = n_p = Mn_{\mathrm{GJ}}$, where M is the multiplicity [19, 23]. Moderate estimates then give $M = 10$ [23]. Choosing this value and letting $\tau = 1\,\mathrm{s}$, we note that for magnetar field strengths, $B_{\mathrm{pulsar}} = 10^{10}\,\mathrm{T}$, the term due to the plasma $\propto \omega_{\mathrm{p}}^2/\omega^2$ in (13) dominates over the term due to QED $\propto 14\xi$ for frequencies up to $\omega \sim 10^{14} - 10^{15}\,\mathrm{rad\,s^{-1}}$, i.e. in the infrared regime and below. Furthermore, we note that photon splitting [15, 16] as described by standard QED (i.e. with zero plasma density) requires that the phase velocity of the dispersion relation in (12) is higher than that of (13). While this is always true in the absence of a plasma, we note that for wave frequencies in the infra-red regime and below, the

Goldreich–Julian density given by (14) is enough to increase the phase velocity of the mode in (13) above that of (12), unless we choose the period time τ extremely low. Thus we conclude that photon–photon splitting as described by vacuum theories is not likely to apply to magnetar atmospheres, unless the pair-production [19] responsible for the Goldreich-Julian expression is effectively suppressed. Wave cascade processes as a mechanism to explain the radio silence of magnetars [18] could still be possible, but for densities of the order of (14), plasma nonlinearities are likely to dominate over the pure QED effects.

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